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A high-order structure-preserving difference scheme for generalized fractional Schrödinger equation with wave operator

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Abstract

This paper focuses on the construction and analysis of the structure-preserving algorithm for generalized fractional Schrödinger equation with wave operator. A fourth-order energy-conserving difference scheme is developed for the resulting equivalent system based on scalar auxiliary variable approach. The discrete energy conservation law, boundedness and convergence of difference solutions are proved in detail. Numerical experiments are performed to verify our theoretical analysis results.

Keywords: Fractional Schrödinger equation with wave operator; Scalar auxiliary variable approach; Structure-preserving algorithm; Energy-conserving scheme; Boundedness; Convergence

1. Introduction

In this paper, we consider the following nonlinear fractional Schrödinger equation with wave operator

$$u_{tt} + (-\Delta)^{\alpha/2}u + iu_t + \beta f(|u|^2)u = 0, \;\; x \in \mathbb{R}, t \in (0, T], \tag{1.1}$$

$$u(x, 0) = \phi_0(x), u_t(x, 0) = \phi_1(x), \;\; x \in \mathbb{R}, \tag{1.2}$$

$$u(x, t) = 0, \;\; x \in \mathbb{R}\backslash\Omega, t \in [0, T], \tag{1.3}$$

where $i = \sqrt{-1}$, $1 < \alpha \leq 2$, and β is a positive constant, $\phi_0(x)$ and $\phi_1(x)$ are known smooth functions, $u(x, t)$ is the complex-valued wave function to be determined, the nonlinear term f is a given real function, $-(-\Delta)^{\alpha/2}u$ denotes the Riesz fractional derivative with order α, see [20].

As well known, many continuous systems possess some physical quantities that naturally arise from the physical context, such as energy, momentum and mass. Therefore, many researchers have devoted themselves to constructing

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numerical methods for preserving the inherent invariants of the original system as much as possible. In terms of this model (1.1)–(1.3), which preserves the energy conservation law as follows

$$E(t) = \|u_t(\cdot,t)\|^2 + \|(-\Delta)^{\alpha/4}u(\cdot,t)\|^2 + \frac{\beta}{2}\int_{\mathbb{R}} F(|u|^2)dx, \tag{1.4}$$

where $F(s) = \int_0^s f(z)dz$, see [1,9]. When $\alpha = 2$, this model (1.1)–(1.3) is simplified to the classical Schrödinger equation with wave operator. Over the past three decades, a great deal of research results has been developed on the classical equations. For example, Bao et al. [1] proposed a difference method and studied the uniform error estimates, Li et al. [7] constructed a compact difference scheme, Wang et al. [16] given discrete-time orthogonal spline collocation method, Guo et al. [5] developed an energy conserving local discontinuous Galerkin method.

With the development of fractional calculus, researchers pay more and more attention to the study of fractional models. For example, Ran and Zhang [9] developed a three-level linearly implicit conservative difference scheme for this model (1.1)–(1.3), Cheng and Wu [3] proposed several conservative compact difference schemes for the same model. Some energy-preserving difference schemes for the single and coupled fractional nonlinear Schrödinger equations without wave operator has also been proposed, see [8,13,15].

However, most of the energy-preserving schemes mentioned above are either completely implicit or multi-layered and low accuracy. A few are constructed with a linear energy-preserving scheme with high-order accuracy for fractional models. The aim of this paper is to develop an energy-preserving and linear numerical scheme with higher-order accuracy for this problem (1.1)–(1.3) based on triangular scalar auxiliary variable (T-SAV) approach proposed in recent years. The ideas and application of SAV approaches, please refer to [4,10–12,18,19]. The advantage of SAV approaches is that nonlinear terms in equations can be treated semi-explicitly. Compared with traditional SAV approach, T-SAV approach adopted in this paper inherits all the advantages of SAV approach, and overcomes some disadvantages such as the nonlinear free energy potential must be bounded from below and an inner product must be calculated to calculate u^{n+1}, see [19].

This article is organized as follows. In Section 2, we introduce the T-SAV approach to rewrite the model (1.1)–(1.3) as an equivalent form. In Section 3, we construct a fourth-order difference scheme for the resulting equivalent system and prove the discrete energy conservation law. In Section 4, the boundedness and convergence of the difference solutions are discussed. In Section 5, some numerical experiments are given to verify the theoretical results.

2. Equivalent system

In order to construct an energy-preserving and linear numerical scheme for solving the problem (1.1)–(1.3), it is necessary to rewrite this problem into its equivalent system by using the T-SAV approach. To do this, let us introduce a triangular scalar auxiliary variable as

$$r(t) = \sin(E_1(u)) + \varepsilon,$$

where

$$E_1(u) = \int_{\Omega} F(|u|^2)dx,$$

and ε is a constant. To ensure that the value of r remains positive, ε has to be greater than 1. Moreover, we usually choose an ε large enough such that $r(t)$ is much greater than zero to avoid singularity, see [19].

As a result, we have

$$\frac{dr}{dt} = \cos(E_1(u))\frac{dE_1(u)}{dt} = \cos(E_1(u))\frac{d}{dt}\int_{\Omega} F(|u|^2)dx. \tag{2.1}$$

Noticing that $\sin^2(E_1(u)) + \cos^2(E_1(u)) = 1$, we take

$$\cos(E_1(u)) = \sqrt{1-\sin^2(E_1(u))} = \sqrt{1-(r-\varepsilon)^2}. \tag{2.2}$$

It follows from (2.1) that

$$\frac{1}{\sqrt{1-(r-\varepsilon)^2}}\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{dE_1(u)}{dt} = \frac{d}{dt}\int_{\Omega} F(|u|^2)dx. \tag{2.3}$$

Taking the derivative of the arcsine function we get

$$\frac{d}{dt}(\arcsin(r-\varepsilon)) = \frac{1}{\sqrt{1-(r-\varepsilon)^2}}\frac{\mathrm{d}r}{\mathrm{d}t}. \tag{2.4}$$

Substituting (2.3) into the right side of (2.4), and we get immediately that

$$\frac{d}{dt}(\arcsin(r-\varepsilon)) = \frac{dE_1(u)}{dt} = \frac{d}{dt}\int_\Omega F(|u|^2)dx = \int_\Omega f(|u|^2)(u\bar{u}_t + u_t\bar{u})dx = 2\mathrm{Re}(\int_\Omega f(|u|^2)u\bar{u}_t dx).$$

Thus Eq. (1.1) can be transformed into an equivalent system as follows

$$u_t = v, \tag{2.5}$$

$$v_t = -(-\Delta)^{\alpha/2}u - iv - b(u)r, \tag{2.6}$$

$$\frac{d}{dt}\arcsin(r-\varepsilon) = \frac{2}{\beta}\mathrm{Re}(b(u)r, u_t), \tag{2.7}$$

where $b(u) = \frac{\beta f(|u|^2)u}{\sin(E_1(u))+\varepsilon}$.

For the resulting equivalent system (2.5)–(2.7), we have the following result.

Theorem 2.1. *The equivalent system (2.5)–(2.7) possess modified energy conservation law as follows*

$$E(t) = E(0), \tag{2.8}$$

where

$$E(t) = \|u_t\|^2 + \|(-\Delta)^{\alpha/4}u\|^2 + \beta\arcsin(r-\varepsilon) = \|v\|^2 + \|(-\Delta)^{\alpha/4}u\|^2 + \beta\arcsin(r-\varepsilon). \tag{2.9}$$

Proof. Computing the inner product of (2.6) with u_t and taking the real part derives

$$\mathrm{Re}(v_t, u_t) = -\mathrm{Re}((-\Delta)^{\alpha/2}u, u_t) - \mathrm{Re}(iv, u_t) - \mathrm{Re}(b(u)r, u_t). \tag{2.10}$$

Noticing that

$$\mathrm{Re}(v_t, u_t) = \frac{1}{2}\frac{d}{dt}\|u_t\|^2,\ \ \mathrm{Re}((-\Delta)^{\alpha/2}u, u_t) = \frac{1}{2}\frac{d}{dt}\|(-\Delta)^{\alpha/4}u\|^2,$$

and (2.7), we can obtain

$$\frac{dE(t)}{dt} = 0.$$

That is,

$$E(t) = E(0).$$

The second equation in (2.9) can be obtained by using (2.5). This proof is completed.

3. High-order energy-conserving difference scheme

In this section, we will first establish a high-order difference scheme and then analyze the discrete energy conservation property of the proposed scheme.

3.1. Preparation

Let $\mathbb{Z}$ be integer field. Denote $x_j = jh$ for $j \in \mathbb{Z}$ and $h\mathbb{Z} = \{jh \mid j \in \mathbb{Z}\}$. For any grid functions u, v on $h\mathbb{Z}$, define the discrete inner product and norms as

$$(u, v) = h\sum_{j\in\mathbb{Z}} u_j\bar{v}_j, \quad \|u\| = \sqrt{(u,u)}, \quad \|u\|_\infty = \sup_{j\in\mathbb{Z}}|u_j|.$$

Set $L_h^2 = \{u_j^n \in \mathbb{C} : \|u^n\| < +\infty\}$, where $\mathbb{C}$ denotes complex field. For $u^n \in L_h^2$, we denote the discrete Fourier transform $\widehat{u^n} : [-\pi, \pi]$ by

$$\hat{u}^n(\omega) = \frac{1}{\sqrt{2\pi}} \sum_{j=-\infty}^{+\infty} u_j^n e^{-ij\omega}.$$

Moreover, we have the inversion formula

$$u_j^n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/h}^{+\pi/h} \hat{u}^n(\omega)\overline{\hat{v}^n(\omega)} d\omega,$$

and Parseval's theorem

$$(u, v) = \int_{-\pi/h}^{+\pi/h} \hat{u}^n(\omega)\overline{\hat{v}^n(\omega)} d\omega.$$

Also, for constant $0 \le \delta \le 1$ and $u \in L_h^2$, define fractional Sobolev norm $\|u\|_{H^\delta}$ and semi-norm $|u|_{H^\delta}$ as

$$\|u\|_{H^\delta}^2 = h \int_{-\pi}^{\pi} (1 + |\omega|^{2\delta}) |\hat{u}(\omega)|^2 d\omega \ , \ |u|_{H^\delta}^2 = h \int_{-\pi}^{\pi} |\omega|^{2\delta} |\hat{u}(\omega)|^2 d\omega.$$

From Parseval's theorem, it is clear that $\|u\|_{H^\delta}^2 = \|u\|^2 + |u|_{H^\delta}^2$ and $|u|_{H^0}^2 = \|u\|^2$.

The following lemmas are also crucial in the derivation and analysis of the difference scheme.

Lemma 3.1 (*Discrete Uniform Sobolev Inequality, [14]*). *For every $1/2 < \delta \le 1$, there exists a constant $C_\delta > 0$ independent of h such that*

$$\|u\|_\infty \le C_\delta \|u\|_{H^\delta}.$$

Lemma 3.2 (*[17]*). *Suppose*

$$u \in \mathscr{C}^{4+\alpha}(\mathbb{R}) = \{v | v \in L^1(\mathbb{R}), \int_{-\infty}^{\infty} (1 + |\varpi|)^{4+\alpha} |\hat{v}(\varpi)\ | \, d\varpi < \infty\},$$

where $\widehat{v}(\varpi)$ is the Fourier transform with respect to $v(x)$, i.e.,

$$\widehat{v}(\varpi) = \int_{-\infty}^{\infty} v(x) e^{-\mathbf{i}\varpi x} dx.$$

Then, we have

$$(-\Delta)^{\alpha/2} u(x) = \frac{1}{h^\alpha} \sum_{k=-\infty}^{+\infty} \widehat{g}_k^{(\alpha)} u(x - kh) + O(h^4) = \delta_h^\alpha u(x) + O(h^4), \tag{3.1}$$

where

$$\widehat{g}_k^{(\alpha)} = \begin{cases} \frac{4}{3} g_k^{(\alpha)} - \frac{1}{3\cdot 2^\alpha} g_{\frac{k}{2}}^{(\alpha)}, k \text{ is even}, \\ \frac{4}{3} g_k^{(\alpha)}, k \text{ is odd}, \end{cases}$$

in which

$$g_k^{(\alpha)} = \left(1 - \frac{\alpha + 1}{\alpha/2 + k}\right) g_{k-1}^{(\alpha)} \text{ and } g_0^{(\alpha)} = \frac{\Gamma(\alpha + 1)}{\Gamma^2(\alpha/2 + 1)}, k = 1, 2, \ldots.$$

3.2. Derivation of the difference scheme

Without loss of generality, denote $\Omega = (a, b)$. Let $\tau = T/N$ be the temporal step size and $h = (b - a)/M$ be spatial step size, where N and M are given positive integers. Denote $t_n = n\tau (0 \le n \le N), x_j = a + jh (0 \le j \le M), \Omega_\tau = \{t_n \mid 0 \le n \le N\}$ and $\Omega_h = \{x_j \mid 0 \le j \le M\}$. Define the grid function space $V_h = \{v = \{v_j\} \mid 0 \le j \le M\}$ and denote $V_h^0 = \{v \mid v \in V_h \text{ and } v_0 = v_M = 0\}$. For any grid function $v^n \in V_h$, we denote

$$\delta_t v_j^{n+\frac{1}{2}} = \frac{v_j^{n+1} - v_j^n}{\tau}, \quad \delta_x v_j^n = \frac{v_{j+1}^n - v_j^n}{h}, \quad v_j^{n+\frac{1}{2}} = \frac{v_j^{n+1} + v_j^n}{2}, \quad \tilde{v}_j^{n+\frac{1}{2}} = \frac{3v_j^n - v_j^{n-1}}{2}.$$

Using Taylor's expansion, and applying the fourth-order fractional central difference approximation (3.1) in space and Crank–Nicolson method in time to equivalent system (2.5)–(2.7) gives that

$$\delta_t U_j^{n+\frac{1}{2}} = V_j^{n+\frac{1}{2}} + T_1^n, \tag{3.2}$$

$$\delta_t V_j^{n+\frac{1}{2}} = -\delta_h^\alpha U_j^{n+\frac{1}{2}} - iV_j^{n+\frac{1}{2}} - b(\tilde{U}^{n+\frac{1}{2}})\tilde{R}^{n+\frac{1}{2}} + T_2^n, \tag{3.3}$$

$$\delta_t \arcsin(R^{n+\frac{1}{2}} - \varepsilon) = \frac{2}{\beta}\mathrm{Re}(b(\tilde{U}^{n+\frac{1}{2}})\tilde{R}^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}) + T_3^n, \tag{3.4}$$

where there is a positive constant C_R such that

$$\max\{|T_1^n|, |\delta_t T_1^n|, |T_3^n|\} \le C_R\tau^2, \max\{|T_2^n|\} \le C_R(\tau^2 + h^4), 0 \le n \le N, \tag{3.5}$$

with $\tilde{U}_j^{n+\frac{1}{2}} = (3U_j^n - U_j^{n-1})/2$ for $n \ge 1$, but $\tilde{U}_j^{\frac{1}{2}} = U_j^0$.

Omitting the error terms in (3.2)–(3.4), and replacing U, V and R by u, v and r respectively, we can obtain the finite difference scheme which reads as

$$\delta_t u_j^{n+\frac{1}{2}} = v_j^{n+\frac{1}{2}}, \tag{3.6}$$

$$\delta_t v_j^{n+\frac{1}{2}} = -\delta_h^\alpha u_j^{n+\frac{1}{2}} - iv_j^{n+\frac{1}{2}} - b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, \tag{3.7}$$

$$\delta_t \arcsin(r^{n+\frac{1}{2}} - \varepsilon) = \frac{2}{\beta}\mathrm{Re}(b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}). \tag{3.8}$$

Denote

$$u^n = (u_1^n, u_2^n, \dots, u_{M-1}^n)^T, v^n = (v_1^n, v_2^n, \dots, v_{M-1}^n)^T,$$

the difference scheme (3.6)–(3.8) can be rewritten in the vector form

$$\delta_t u^{n+\frac{1}{2}} = v^{n+\frac{1}{2}}, \tag{3.9}$$

$$\delta_t v^{n+\frac{1}{2}} = -A^{(\alpha)}u^{n+\frac{1}{2}} - iv^{n+\frac{1}{2}} - b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, \tag{3.10}$$

$$\delta_t \arcsin(r^{n+\frac{1}{2}} - \varepsilon) = \frac{2}{\beta}\mathrm{Re}(b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}), \tag{3.11}$$

where

$$A^{(\alpha)} = \frac{1}{h^\alpha}\begin{bmatrix} \widehat{g}_0^{(\alpha)} & \widehat{g}_{-1}^{(\alpha)} & \widehat{g}_{-2}^{(\alpha)} & \cdots & \widehat{g}_{-M+4}^{(\alpha)} & \widehat{g}_{-M+3}^{(\alpha)} & \widehat{g}_{-M+2}^{(\alpha)} \\ \widehat{g}_1^{(\alpha)} & \widehat{g}_0^{(\alpha)} & \widehat{g}_{-1}^{(\alpha)} & \cdots & \widehat{g}_{-M+5}^{(\alpha)} & \widehat{g}_{-M+4}^{(\alpha)} & \widehat{g}_{-M+3}^{(\alpha)} \\ \widehat{g}_2^{(\alpha)} & \widehat{g}_1^{(\alpha)} & \widehat{g}_0^{(\alpha)} & \cdots & \widehat{g}_{-M+6}^{(\alpha)} & \widehat{g}_{-M+5}^{(\alpha)} & \widehat{g}_{-M+4}^{(\alpha)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \widehat{g}_{M-4}^{(\alpha)} & \widehat{g}_{M-5}^{(\alpha)} & \widehat{g}_{M-6}^{(\alpha)} & \cdots & \widehat{g}_0^{(\alpha)} & \widehat{g}_{-1}^{(\alpha)} & \widehat{g}_{-2}^{(\alpha)} \\ \widehat{g}_{M-3}^{(\alpha)} & \widehat{g}_{M-4}^{(\alpha)} & \widehat{g}_{M-5}^{(\alpha)} & \cdots & \widehat{g}_1^{(\alpha)} & \widehat{g}_0^{(\alpha)} & \widehat{g}_{-1}^{(\alpha)} \\ \widehat{g}_{M-2}^{(\alpha)} & \widehat{g}_{M-3}^{(\alpha)} & \widehat{g}_{M-4}^{(\alpha)} & \cdots & \widehat{g}_2^{(\alpha)} & \widehat{g}_1^{(\alpha)} & \widehat{g}_0^{(\alpha)} \end{bmatrix}.$$

It is worth noting that $A^{(\alpha)}$ corresponding to the operator δ_h^α is symmetric Toeplitz matrix since the coefficients $\widehat{g}_k^{(\alpha)}$ defined in Lemma 3.2 satisfy that [17]

$$\widehat{g}_0^{(\alpha)} > 0,\ \widehat{g}_k^{(\alpha)} = \widehat{g}_{-k}^{(\alpha)} < 0,\ \sum_{k=-\infty}^{+\infty} \widehat{g}_k^{(\alpha)} = 0.$$

The above equivalent form (3.9)–(3.11) is only used in programming implementation.

3.3. Discrete energy conservation law

In this subsection, we focus on the conservation of the difference scheme (3.6)–(3.8), and the following lemmas are firstly introduced.

Lemma 3.3 ([17]). *For any grid functions $v^n \in V_h^0$, there exists a linear operator $L^{(\alpha)}$ such that*

$$Im(\delta_h^\alpha v^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) = 0,\ Re(\delta_h^\alpha v^{n+\frac{1}{2}}, \delta_t v^{n+\frac{1}{2}}) = \frac{1}{2\tau}(\|L^{(\alpha)}v^{n+1}\|^2 - \|L^{(\alpha)}v^n\|^2).$$

Based on the previous preparation, we can prove the following result.

Theorem 3.4. *The difference scheme (3.6)–(3.8) is energy-conserving, i.e., it satisfies that*

$$E^n = E^0, 1 \le n \le N,$$

where

$$E^n = \|v^n\|^2 + \|L^{(\alpha)}u^n\|^2 + \beta \arcsin(r^n - \varepsilon).$$

Proof. Computing the inner product of (3.6) with $\delta_t v^{n+\frac{1}{2}}$, and taking the real part, we have

$$\mathrm{Re}(\delta_t u^{n+\frac{1}{2}}, \delta_t v^{n+\frac{1}{2}}) = \mathrm{Re}(v^{n+\frac{1}{2}}, \delta_t v^{n+\frac{1}{2}}) = \frac{1}{2\tau}(\|v^{n+1}\|^2 - \|v^n\|^2). \tag{3.12}$$

Similarly, computing the inner product of (3.7) with $\delta_t u^{n+\frac{1}{2}}$ and taking the real part, we obtain

$$\mathrm{Re}(\delta_t v^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}) = -\mathrm{Re}(\delta_h^\alpha u^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}) - \mathrm{Re}(i v^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}) - \mathrm{Re}(b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}). \tag{3.13}$$

Using Lemma 3.3, we have

$$\mathrm{Re}(\delta_h^\alpha u^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}) = \frac{1}{2\tau}(\|L^{(\alpha)}u^{n+1}\|^2 - \|L^{(\alpha)}u^n\|^2). \tag{3.14}$$

Also, it follows from (3.6) that

$$\mathrm{Re}(i v^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}}) = \mathrm{Re}(i v^{n+\frac{1}{2}}, v^{n+\frac{1}{2}}) = 0. \tag{3.15}$$

Combining (3.12)–(3.14) with (3.8), it holds that

$$E^{n+1} = E^n,\ 1 \le n \le N,$$

where

$$E^n = \|v^n\|^2 + \|L^{(\alpha)}u^n\|^2 + \beta \arcsin(r^n - \varepsilon).$$

It completes the proof.

4. Boundedness and convergence

In this section, let us turn our attention to the boundedness and convergence of solutions of the difference scheme (3.6)–(3.8).

4.1. Boundedness

The following lemmas are important tools in proving the boundedness of difference solution.

Lemma 4.1 ([17]). *For $1 < \alpha \le 2$, we have*

$$\left(\frac{4}{3}|\frac{2}{\pi}|^\alpha - \frac{1}{3}\right)|u^n|_{H^{\alpha/2}} \le (\delta_h^\alpha u^n, u^n) \le \left(\frac{4}{3} - \frac{1}{3}|\frac{2}{\pi}|^\alpha\right)|u^n|_{H^{\alpha/2}}.$$

Lemma 4.2 ([6]). *For time sequences $w = \{w^0, w^1, \dots, w^n\}$ and $g = \{g^0, g^1, \dots, g^n\}$, there is*

$$|2\tau \sum_{k=0}^{n} g^k \delta_t w^{k+\frac{1}{2}}| \le \tau \sum_{k=1}^{n} |w^k|^2 + \tau \sum_{k=0}^{n-1} |\delta_t g^{k+\frac{1}{2}}|^2 + \frac{1}{2}|w^{n+1}|^2 + 2|g^n|^2 + |w^0|^2 + |g^0|^2.$$

Lemma 4.3 (*Gronwall inequality I, [21]*). *Suppose that the discrete grid function $\{w^n \mid n = 0, 1, \dots, N; N\tau = T\}$ satisfies the following inequality*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n\tau,$$

where A, B and C_n are non-negative constants, then

$$\max_{1\le n\le N} |w^n| \le \left(w^0 + \tau \sum_{k=1}^{N} C_k \right) e^{2(A+B)T},$$

where τ is sufficiently small, such that $(A + B)\tau \le \frac{N-1}{2N} < \frac{1}{2}(N > 1)$.

Lemma 4.4 (*Gronwall inequality II, [21]*). *Suppose that the discrete grid function $\{w^n \mid n = 0, 1, \dots, N\}$ satisfies the following inequality*

$$w^n \le A + \tau \sum_{k=1}^{n} B_k w^k,$$

where A and B_k are non-negative constants, then we have

$$\max_{1\le n\le N} |w^n| \le A \exp\left(2\tau \sum_{k=1}^{N} B_k \right),$$

where τ is sufficiently small, such that $\tau \max_{1\le k\le N} B_k \le 1/2$.

Based on above lemmas, we can prove the following boundedness result.

Theorem 4.5. *Suppose nonlinear energy functional bounded from below, then we have estimates as follows:*

$$\|u^n\|_\infty \le C, \ \ 1 \le n \le N,$$

where C is a positive constant which is independent of τ and h.

Proof. It follows from Theorem 3.4, there exists a constant C such that

$$E^n = \|v^n\|^2 + \|L^{(\alpha)}u^n\|^2 + \beta \arcsin(r^n - \varepsilon) = E^0 = C.$$

It implies that

$$\|v^n\| \le C, \|L^{(\alpha)}u^n\| \le C.$$

Computing the inner product of (3.9) with $u^{n+\frac{1}{2}}$, taking the real part, gives that

$$\frac{1}{2\tau}(\|u^{n+1}\|^2 - \|u^n\|^2) = \mathrm{Re}(v^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}) \le \|v^{n+\frac{1}{2}}\|(\|u^{n+1}\| + \|u^n\|)/2 \le C(\|u^{n+1}\| + \|u^n\|)/2.$$

That is,

$$\|u^{n+1}\| - \|u^n\| \le C\tau.$$

Noticing that $n\tau \le T$, and summing up the above inequality for n from 0 to n yields

$$\|u^n\| \le C.$$

From Lemma 3.1, there is a positive constant $\delta \in (1/2, 1]$ such that

$$\|u\|_\infty^2 \le C_\delta^2 \|u\|_{H^\delta}^2 = C_\delta^2(\|u\|^2 + |u|_{H^\delta}^2).$$

Noticing that $(\delta_h^\alpha u^n, u^n) = \|L^{(\alpha)}u^n\|^2$, using Lemma 4.1 yields that

$$\|u\|_\infty^2 \le C_\delta^2 \left(C + \frac{(\delta_h^\alpha u^n, u^n)^2}{(\frac{4}{3}|\frac{2}{\pi}|^{2\delta} - \frac{1}{3})^2} \right) = C_\delta^2 \left(C + \frac{\|L^{(\alpha)}u^n\|^4}{(\frac{4}{3}|\frac{2}{\pi}|^{2\delta} - \frac{1}{3})^2} \right) \le C.$$

This proof is completed.

4.2. Convergence

Now we turn to the convergence of solutions of the difference scheme (3.8)–(3.10).

Let $e_j^n = U_j^n - u_j^n$, then we can obtain the following convergence results.

Theorem 4.6. *Assuming that the solution of the problem (1.1)–(1.3) is sufficiently smooth, then the solution of difference scheme (3.6)–(3.8) satisfies that*

$$\|e^n\|_\infty \le C(\tau^2 + h^4),\ \ 1 \le n \le N.$$

when $\tau < tau_0$, and C and τ_0 are positive constants independent of τ and h.

Proof. Subtracting (3.6)–(3.8) from (3.2)–(3.4), we obtain error system as follows

$$\delta_t e^{n+\frac{1}{2}} = \eta^{n+\frac{1}{2}} + T_1^n, \tag{4.1}$$

$$\delta_t \eta^{n+\frac{1}{2}} = -\delta_h^\alpha e^{n+\frac{1}{2}} - i\eta^{n+\frac{1}{2}} - (b(\tilde{U}^{n+\frac{1}{2}})\tilde{R}^{n+\frac{1}{2}} - b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}) + T_2^n, \tag{4.2}$$

$$\delta_t \zeta^{n+\frac{1}{2}} = \frac{2}{\beta}[\mathrm{Re}(b(\tilde{U}^{n+\frac{1}{2}})\tilde{R}^{n+\frac{1}{2}}, \delta_t U^{n+\frac{1}{2}}) - \mathrm{Re}(b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, \delta_t u^{n+\frac{1}{2}})] + T_3^n, \tag{4.3}$$

where $\eta^n = V^n - v^n$, $\zeta^n = \arcsin(R^n - \varepsilon) - \arcsin(r^n - \varepsilon)$.

(I) First we consider the convergence result when $n = 0$. From (1.2), we get

$$\|e^0\| = 0,\ \ \|\eta^0\| = 0. \tag{4.4}$$

Computing the inner product of (4.1) with $\delta_t \eta^{\frac{1}{2}}$, and taking the real part, we obtain

$$\mathrm{Re}(\delta_t e^{\frac{1}{2}}, \delta_t \eta^{\frac{1}{2}}) = \mathrm{Re}(\eta^{\frac{1}{2}}, \delta_t \eta^{\frac{1}{2}}) + \mathrm{Re}(T_1^0, \delta_t \eta^{\frac{1}{2}}) = \frac{1}{2\tau}\|\eta^1\|^2 + \frac{1}{\tau}\mathrm{Re}(T_1^0, \eta^1). \tag{4.5}$$

Similarly, computing the inner product of (4.2) with $\delta_t e^{\frac{1}{2}}$, and taking the real part, we have

$$\begin{aligned}\mathrm{Re}(\delta_t \eta^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) &= -\mathrm{Re}(\delta_h^\alpha e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) - \mathrm{Re}(i\eta^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) - \mathrm{Re}(G^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + \mathrm{Re}(T_2^0, \delta_t e^{\frac{1}{2}})\\ &= -\frac{1}{2\tau}\|L^{(\alpha)}e^1\|^2 - \mathrm{Re}(i\eta^{\frac{1}{2}}, \eta^{\frac{1}{2}} + T_1^0) - \mathrm{Re}(G^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + \mathrm{Re}(T_2^0, \delta_t e^{\frac{1}{2}}),\end{aligned} \tag{4.6}$$

where

$$G^{\frac{1}{2}} = g(\tilde{U}^{\frac{1}{2}}) - g(\tilde{u}^{\frac{1}{2}})$$

with $g(u) = \beta f(|u|^2)u$. According to the continuity of the function g, we have

$$G^{\frac{1}{2}} = g(\tilde{U}^{\frac{1}{2}}) - g(\tilde{u}^{\frac{1}{2}}) = g'(\xi^n)(\tilde{U}^{\frac{1}{2}} - \tilde{u}^{\frac{1}{2}}) \le C_1|e^1|,$$

where $C_1 = \frac{3}{2}\max|g'(\xi^n)|$ and ξ^n is on the segment that connects U^n and u^n.

Moreover, by using (4.1) and (4.4), it follows that

$$\|e^1\| \le \frac{\tau}{2}\|\eta^1\| + \tau\|T_1^0\|. \tag{4.7}$$

This combining with (4.5)–(4.6) and $C_2 > 2C_1$ gives that

$$\begin{aligned}&\|\eta^1\|^2 + \|L^{(\alpha)}e^1\|^2\\ =&-2\mathrm{Re}(T_1^0, \eta^1) - \tau\mathrm{Re}(i\eta^1, T_1^0) + 2\mathrm{Re}(T_2^0, e^1) - 2\mathrm{Re}(G^{\frac{1}{2}}, e^1)\\ \le&\ 2\mathrm{Re}(T_1^0, \eta^1) + \tau\mathrm{Re}(i\eta^1, T_1^0) + 2\mathrm{Re}(T_2^0, e^1) + 2C_2\|e^1\|^2\\ \le&\ [(2+\tau)\tau\|T_1^0\| + \tau\|T_2^0\|]^2 + \frac{1}{4}\|\eta^1\|^2 + 2\tau\|T_1^0\|\|T_2^0\| + \frac{3C_2\tau^2}{2}\|\eta^1\|^2 + 3C_2\tau^2\|T_1^0\|^2.\end{aligned}$$

Let $C_2\tau^2 \le 1/3$, by virtue of (3.5), we have

$$\begin{aligned}
&\|\eta^1\|^2 + \|L^{(\alpha)}e^1\|^2\\
&\le [(2+\tau)\tau\|T_1^0\| + \tau\|T_2^0\|]^2 + \frac{3}{4}\|\eta^1\|^2 + 2\tau\|T_1^0\|\|T_2^0\| + \|T_1^0\|^2\\
&\le 4[(2+\tau)\sqrt{(b-a)}C_R\tau^2 + \tau\sqrt{(b-a)}C_R(\tau^2+h^4)]^2 + 2\tau(b-a){C_R}^2\tau^2(\tau^2+h^4) + (b-a){C_R}^2\tau^4\\
&\le C_3(\tau^2+h^4)^2,
\end{aligned}$$

where $C_3 = [26T + 16T^2 + 17](b-a)C_R^2$. It implies that

$$\|\eta^1\| \le C_4(\tau^2+h^4),\ \|L^{(\alpha)}e^1\| \le C_4(\tau^2+h^4). \tag{4.8}$$

where $C_4 = \sqrt{C_3}$. According to the estimate in (4.7), we obtain

$$\|e^1\| \le C_5(\tau^2+h^4), \tag{4.9}$$

where $C_5 = (\frac{1}{2}C_4 + \sqrt{b-a}C_R)T$.

Applying Lemmas 3.1 and 4.1 and combining the second inequality in (4.8), we have

$$\|e^1\|_\infty \le C_6(\tau^2+h^4), \tag{4.10}$$

where $C_6 = C_\delta\sqrt{C_5^2 + \frac{C_4^2}{(\frac{4}{3}|\frac{2}{\pi}|^{2\delta}-\frac{1}{3})^2}}$ and δ is an arbitrary constant between $1/2$ and 1.

(II) Now we consider the convergence for $n \geqslant 1$.

Computing the inner product of (4.1) with $\delta_t\eta^{n+\frac{1}{2}}$, and taking the real part yields that

$$\begin{aligned}
\mathrm{Re}(\delta_t e^{n+\frac{1}{2}}, \delta_t\eta^{n+\frac{1}{2}}) &= \mathrm{Re}(\eta^{n+\frac{1}{2}}, \delta_t\eta^{n+\frac{1}{2}}) + \mathrm{Re}(T_1^n, \delta_t\eta^{n+\frac{1}{2}})\\
&= \frac{1}{2\tau}(\|\eta^{n+1}\|^2 - \|\eta^n\|^2) + \mathrm{Re}(T_1^n, \delta_t\eta^{n+\frac{1}{2}}).
\end{aligned} \tag{4.11}$$

Computing the discrete inner product of (4.2) with $\delta_t e^{n+\frac{1}{2}}$, and taking the real part gives that

$$\begin{aligned}
&\mathrm{Re}(T_2^n, \delta_t e^{n+\frac{1}{2}})\\
&= \mathrm{Re}(\delta_t\eta^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}) + \mathrm{Re}(\delta_h^\alpha e^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}) + \mathrm{Re}(i\eta^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}) + \mathrm{Re}(G^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}).
\end{aligned} \tag{4.12}$$

where $G^{n+\frac{1}{2}} = b(\tilde{U}^{n+\frac{1}{2}})\tilde{R}^{n+\frac{1}{2}} - b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}$.

Combining (4.11)–(4.1) and (4.12), we have

$$\begin{aligned}
&\mathrm{Re}(T_2^n, \delta_t e^{n+\frac{1}{2}}) - \mathrm{Re}(T_1^n, \delta_t\eta^{n+\frac{1}{2}})\\
&= \frac{1}{2\tau}(\|\eta^{n+1}\|^2 - \|\eta^n\|^2) + \frac{1}{2\tau}(\|L^{(\alpha)}e^{n+1}\|^2 - \|L^{(\alpha)}e^n\|^2) + \mathrm{Re}(i\eta^{n+\frac{1}{2}}, T_1^n) + \mathrm{Re}(G^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}).
\end{aligned} \tag{4.13}$$

According to the continuity of the function b, we have

$$b(\tilde{U}^{n+\frac{1}{2}}) - b(\tilde{u}^{n+\frac{1}{2}}) = b'(\xi^n)(\tilde{U}^{n+\frac{1}{2}} - \tilde{u}^{n+\frac{1}{2}}) \le C_7(|e^n| + |e^{n-1}|),$$

where $C_7 = \frac{3}{2}\max|b'(\xi^n)|$ and ξ^n is on the segment that connects U^n and u^n.

From (3.11), we have

$$r^{n+1} = \sin(\frac{2}{\beta}\mathrm{Re}(b(\tilde{u}^{n+\frac{1}{2}})\tilde{r}^{n+\frac{1}{2}}, u^{n+1} - u^n) + \arcsin(r^n - \varepsilon)) + \varepsilon.$$

It means that

$$|\frac{\varepsilon-1}{\varepsilon+1}| \le |\frac{\tilde{r}^{n+\frac{1}{2}}}{\sin(E_1(\tilde{u}^{n+\frac{1}{2}})) + \varepsilon}| \le |\frac{\varepsilon+1}{\varepsilon-1}|.$$

Based on the boundedness in Theorem 4.5, we further have

$$|(1 - \frac{2}{\varepsilon+1})| \cdot |b(\tilde{u}^{n+\frac{1}{2}})| \le \frac{|\tilde{r}^{n+\frac{1}{2}}| \cdot |b(\tilde{u}^{n+\frac{1}{2}})|}{|\sin(E_1(\tilde{u}^{n+\frac{1}{2}})) + \varepsilon|} \le |(1 + \frac{2}{\varepsilon-1})| \cdot |b(\tilde{u}^{n+\frac{1}{2}})|.$$

As a result, when ε is sufficiently large, we have

$$\frac{\tilde{r}^{n+\frac{1}{2}} b(\tilde{u}^{n+\frac{1}{2}})}{\sin(E_1(\tilde{u}^{n+\frac{1}{2}})) + \varepsilon} \leq C_8 |b(\tilde{u}^{n+\frac{1}{2}})|,$$

where C_8 is a positive constant. Specifically, to trade off the accuracy and efficiency, the parameter $1/\varepsilon$ should be much less than $\tau^2 + h^4$.

It means that

$$\begin{aligned} \|G^{n+\frac{1}{2}}\|^2 &= \left\| \frac{\tilde{R}^{n+\frac{1}{2}} b(\tilde{U}^{n+\frac{1}{2}})}{\sin(E_1(\tilde{U}^{n+\frac{1}{2}})) + \varepsilon} - \frac{\tilde{r}^{n+\frac{1}{2}} b(\tilde{u}^{n+\frac{1}{2}})}{\sin(E_1(\tilde{u}^{n+\frac{1}{2}})) + \varepsilon} \right\|^2 = \left\| b(\tilde{U}^{n+\frac{1}{2}}) - \frac{\tilde{r}^{n+\frac{1}{2}} b(\tilde{u}^{n+\frac{1}{2}})}{\sin(E_1(\tilde{u}^{n+\frac{1}{2}})) + \varepsilon} \right\|^2 \\ &\leq C_9 \left\| b(\tilde{U}^{n+\frac{1}{2}}) - b(\tilde{u}^{n+\frac{1}{2}}) \right\|^2 \leq C_{10}(\|e^n\|^2 + \|e^{n-1}\|^2), \end{aligned} \tag{4.14}$$

where C_9, C_{10} are some positive constants independent of τ and h.

Combining (4.13)–(4.14) with (4.1), using Cauchy–Schwarz inequality gives that

$$\begin{aligned} &\mathrm{Re}(T_2^n - G^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}) \\ =& \mathrm{Re}(T_2^n - G^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}} + T_1^n) \\ \leq& \|T_2^n - G^{n+\frac{1}{2}}\| \|\eta^{n+\frac{1}{2}} + T_1^n\| \\ \leq& \frac{1}{2}\|T_2^n - G^{n+\frac{1}{2}}\|^2 + \|\eta^{n+\frac{1}{2}}\|^2 + \|T_1^n\|^2 \\ \leq& \|T_2^n\|^2 + \|G^{n+\frac{1}{2}}\|^2 + \frac{1}{2}\|\eta^{n+1}\|^2 + \frac{1}{2}\|\eta^n\|^2 + \|T_1^n\|^2 \\ \leq& \|T_2^n\|^2 + C_{10}(\|e^n\|^2 + \|e^{n-1}\|^2) + \frac{1}{2}\|\eta^{n+1}\|^2 + \frac{1}{2}\|\eta^n\|^2 + \|T_1^n\|^2 \\ \leq& C_{11}(\|e^n\|^2 + \|e^{n-1}\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2 + \|T_1^n\|^2 + \|T_2^n\|^2), \end{aligned} \tag{4.15}$$

where $C_{11} = 1 + C_{10}$. Combining (4.13) and (4.15), we deduce that

$$\begin{aligned} &\frac{1}{2\tau}(\|\eta^{n+1}\|^2 + \|L^{(\alpha)} e^{n+1}\|^2) \\ =& \frac{1}{2\tau}(\|\eta^n\|^2 + \|L^{(\alpha)} e^n\|^2) + \mathrm{Re}(T_2^n - G^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}) - \mathrm{Re}(i\eta^{n+\frac{1}{2}}, T_1^n) - \mathrm{Re}(T_1^n, \delta_t \eta^{n+\frac{1}{2}}) \\ \leq& \frac{1}{2\tau}(\|\eta^n\|^2 + \|L^{(\alpha)} e^n\|^2) + \mathrm{Re}(T_2^n - G^{n+\frac{1}{2}}, \delta_t e^{n+\frac{1}{2}}) - \|T_1^n\| \|\eta^{n+1}\| - \mathrm{Re}(T_1^n, \delta_t \eta^{n+\frac{1}{2}}) \\ \leq& \frac{1}{2\tau}(\|\eta^n\|^2 + \|L^{(\alpha)} e^n\|^2) - \mathrm{Re}(T_1^n, \delta_t \eta^{n+\frac{1}{2}}) + C_{11}(\|e^n\|^2 + \|e^{n-1}\|^2 + \|\eta^{n+1}\|^2 + \|\eta^n\|^2 + \|T_1^n\|^2 + \|T_2^n\|^2). \end{aligned} \tag{4.16}$$

Replacing n by k in (4.16), we can obtain the recurrence by summing up for k from 0 to n as follows

$$\begin{aligned} F^{n+1} \leq& F^0 + C_{12}\tau \sum_{k=0}^{n}(\|e^k\|^2 + \|e^{k-1}\|^2 + \|\eta^{k+1}\|^2 + \|\eta^k\|^2 + \|T_1^k\|^2 + \|T_2^k\|^2) - 2\tau \sum_{k=0}^{n} \mathrm{Re}(T_1^k, \delta_t \eta^{k+\frac{1}{2}}) \\ \leq& C_{13}\tau \sum_{k=0}^{n}(\|e^{k+1}\|^2 + \|\eta^{k+1}\|^2) + C_{12}\tau \sum_{k=0}^{n}(\|T_1^k\|^2 + \|T_2^k\|^2) - 2\tau \sum_{k=0}^{n} \mathrm{Re}(T_1^k, \delta_t \eta^{k+\frac{1}{2}}), \end{aligned} \tag{4.17}$$

where $F^n = \|\eta^n\|^2 + \|L^{(\alpha)} e^n\|^2$, $C_{12} = 2C_{11}$ and $C_{13} = 2C_{12}$.

Computing the inner product of (4.1) with $e^{n+\frac{1}{2}}$, and taking the real part, we get that

$$\begin{aligned} \frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) &= \mathrm{Re}(\eta^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) + \mathrm{Re}(T_1^n, e^{n+\frac{1}{2}}) \\ &\leq \|\eta^{n+\frac{1}{2}}\| \|e^{n+\frac{1}{2}}\| + \|T_1^n\| \|e^{n+\frac{1}{2}}\| \\ &\leq \frac{1}{2}(\|\eta^{n+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2 + \|T_1^n\|^2 + \|e^{n+\frac{1}{2}}\|^2) \end{aligned}$$

$$
\begin{aligned}
&\le \frac{1}{2}(\|e^{n+1}\|^2+\|e^n\|^2)+\frac{1}{4}(\|\eta^{n+1}\|^2+\|\eta^n\|^2)+\frac{1}{2}\|T_1^n\|^2\\
&\le \frac{1}{2}(\|e^{n+1}\|^2+\|e^n\|^2+\|\eta^{n+1}\|^2+\|\eta^n\|^2+\|T_1^n\|^2),
\end{aligned}\tag{4.18}
$$

where the Cauchy–Schwarz inequality has been used. It implies that

$$
\|e^{n+1}\|^2-\|e^n\|^2\le \tau\|e^{n+1}\|^2+\tau\|e^n\|^2+\tau(\|\eta^{n+1}\|^2+\|\eta^n\|^2+\|T_1^n\|^2).\tag{4.19}
$$

Let $0<\tau\le 1/4$, using Lemma 4.3, we can get from (4.19) that

$$
\|e^{n+1}\|^2\le e^{4T}\tau\sum_{k=0}^{n}(\|\eta^{k+1}\|^2+\|\eta^k\|^2+\|T_1^k\|^2)\le C_{14}\tau\sum_{k=0}^{n}\|\eta^{k+1}\|^2+C_{15}(\tau^2)^2,\tag{4.20}
$$

where $C_{14}=2e^{4T}$, $C_{15}=e^{4T}C_R^2(b-a)$.

Thus we have

$$
\begin{aligned}
\tau\sum_{k=0}^{n}\|e^{k+1}\|^2&\le C_{14}\tau^2\sum_{k=0}^{n}\sum_{j=0}^{k}\|\eta^{j+1}\|^2+C_{15}(\tau^2)^2\\
&\le C_{14}\tau^2\sum_{k=0}^{n}\sum_{j=0}^{n}\|\eta^{j+1}\|^2+C_{15}(\tau^2)^2\\
&\le C_{14}T\tau\sum_{k=0}^{n}\|\eta^{k+1}\|^2+C_{15}(\tau^2)^2.
\end{aligned}\tag{4.21}
$$

Combining (4.17) and (4.21) gives that

$$
\begin{aligned}
F^{n+1}\le&F^0+C_{12}\tau\sum_{k=0}^{n}(\|e^k\|^2+\|e^{k-1}\|^2+\|\eta^{k+1}\|^2+\|\eta^k\|^2+\|T_1^k\|^2+\|T_2^k\|^2)-2\tau\sum_{k=0}^{n}(T_1^k,\delta_t\eta^{k+\frac{1}{2}})\\
\le&C_{16}\tau\sum_{k=0}^{n}\|\eta^{k+1}\|^2+C_{13}\tau\sum_{k=0}^{n}(\|T_1^k\|^2+\|T_2^k\|^2)-2\tau\sum_{k=0}^{n}(T_1^k,\delta_t\eta^{k+\frac{1}{2}}),
\end{aligned}\tag{4.22}
$$

where $C_{16}=C_{13}+C_{13}C_{14}T$. Using Lemma 4.2 again, it follows from (4.22) that

$$
|2\tau\sum_{k=0}^{n}(T_1^k,\delta_t\eta^{k+\frac{1}{2}})|\le \tau\sum_{k=1}^{n}\|\eta^k\|^2+\frac{1}{2}\|\eta^{n+1}\|^2+C_{17}(\tau^2)^2.\tag{4.23}
$$

where $C_{17}=2C_R^2(b-a)$. Combining (4.22) with (4.23), we have

$$
\|\eta^{n+1}\|^2+\|L^{(\alpha)}e^{n+1}\|^2\le C_{18}\tau\sum_{k=1}^{n+1}\|\eta^k\|^2+\frac{1}{2}\|\eta^{n+1}\|^2+C_{19}(\tau^2+h^4)^2.
$$

where $C_{18}=1+C_{16}$ and $C_{19}=2C_{13}C_R^2(b-a)T+C_{17}$.

That is,

$$
\begin{aligned}
&\frac{1}{2}\|\eta^{n+1}\|^2+\|L^{(\alpha)}e^{n+1}\|^2\\
\le&C_{18}\tau\sum_{k=1}^{n+1}\|\eta^k\|^2+C_{19}(\tau^2+h^4)^2\\
\le&C_{20}\tau\sum_{k=1}^{n+1}\frac{1}{2}\|\eta^k\|^2+C_{19}(\tau^2+h^4)^2\\
\le&C_{20}\tau\sum_{k=1}^{n+1}(\frac{1}{2}\|\eta^k\|^2+\|L^{(\alpha)}e^k\|^2)+C_{19}(\tau^2+h^4)^2.
\end{aligned}
$$

Table 1
Errors and convergence order in space for Example 5.1 with different values of α and $\tau = 1/10000$.

h	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	$E(h,\tau)$	Ord_1	$E(h,\tau)$	Ord_1	$E(h,\tau)$	Ord_1
1/4	8.7786e−4	–	1.6000e−3	–	2.5000e−3	–
1/8	9.0697e−5	3.2749	1.5739e−4	3.3457	2.5351e−4	3.2967
1/16	5.4139e−6	4.0663	1.0634e−5	3.8875	1.9056e−5	3.7338
1/32	3.3103e−7	4.0317	6.7938e−7	3.9684	1.2465e−6	3.9343
1/64	1.9195e−9	4.1081	4.0709e−8	4.0608	7.7686e−8	4.0040

where $C_{20} = 2C_{18}$. Further, we derive from Lemma 4.4 that

$$\|\eta^{n+1}\|^2 \le C_{21}(\tau^2 + h^4), \;\; \|L^{(\alpha)} e^{n+1}\|^2 \le C_{22}(\tau^2 + h^4), \quad 1 \le n \le N, \tag{4.24}$$

where $C_{21} = 2C_{22} = 2C_{19} e^{2C_{20}T}$. Substituting (4.24) into (4.20) gives that

$$\|e^{n+1}\|^2 \le C_{14}\tau \sum_{k=0}^{n} \|\eta^{k+1}\|^2 + C_{15}(\tau^2)^2 \le C_{23}(\tau^2 + h^4)^2,$$

where $C_{23} = C_{15} + C_{14} C_{21}^2 T$.

Applying Lemmas 3.1 and 4.1 again, we can obtain from above inequality that

$$\|e^n\|_\infty \le C_{24}(\tau^2 + h^4),$$

where $C_{24} = C_\delta \sqrt{C_{23} + \frac{C_{22}^2}{(\frac{4}{3}|\frac{2}{\pi}|^{2\delta} - \frac{1}{3})^2}}$. This proof is completed.

5. Numerical experiments

In this section, we adopt numerical experiments to demonstrate our convergence results and discrete conservation law. Denote calculation error $E(h,\tau) = \max_{1\le n\le N} \|u^n - U^n\|_\infty$, where u^n and U^n denote the exact solution (or reference solution when the analytical solution is unknown) and numerical solution calculated by h and τ at time t_n, respectively.

Also, we define the convergence order in spatial and temporal directions, respectively, by

$$\mathrm{Ord}_1 = \log_2 \frac{E(h,\tau)}{E(h/2,\tau)}, \;\; \mathrm{Ord}_2 = \log_2 \frac{E(h,\tau)}{E(h,\tau/2)}$$

for sufficiently small τ and h, respectively. Based on the previous analysis, to trade off the accuracy and efficiency, the parameter $1/\varepsilon$ should be much less than $\tau^2 + h^4$. Thus we take $\varepsilon = 1.0 \times 10^8$ in the simulations.

Example 5.1. We first consider the following problem with a source term:

$$u_{tt} + (-\Delta)^{\alpha/2} u + i u_t + |u|^2 u = f(x,t), x \in \Omega = [0,1], 0 < t \le 1. \tag{5.1}$$

The initial conditions and source term $f(x,t)$ are determined by exact solution $u(x,t) = (t+1)^3 x^4 (1-x)^4$.

In Tables 1 and 2, we list the errors and convergence orders in spatial and temporal directions, which are obtained by fixing τ and h small enough, respectively. It is clearly observed that the convergence order is close to 4 in space and 2 in time, which is consistent with our theoretical analysis.

For comparison, we calculated the errors and convergence order with respect to the T-SAV scheme (3.6)–(3.8), SAV scheme in [2] and the three-level linearly implicit scheme in [9] for Example 5.1. The results are listed in Table 3 for different h when $\tau = 1/1000$. It is easy to observe that the proposed method in this paper has smaller error and higher convergence order.

It is worth noting that because the source term $f(x,t)$ is not equivalent to zero, the discrete energy conservation law aforementioned in Theorem 3.4 is no longer valid, thus we here do not verify it.

Table 2
Error and convergence order in time for Example 5.1 with different values of α and $h = 1/100$.

τ	$\alpha = 1.2$		$\alpha = 1.5$		$\alpha = 1.8$	
	$E(h,\tau)$	Ord_2	$E(h,\tau)$	Ord_2	$E(h,\tau)$	Ord_2
1/4	3.3069e−4	–	3.5895e−4	–	3.7982e−4	–
1/8	8.3133e−5	1.9920	8.5296e−5	2.0732	9.7551e−5	1.9611
1/16	2.0794e−5	1.9993	2.1011e−5	2.0213	2.5058e−6	1.9609
1/32	5.2029e−6	1.9988	5.2362e−6	2.0046	6.3251e−6	1.9861
1/64	1.3024e−6	1.9981	1.3084e−6	2.0008	1.5876e−6	1.9943

Table 3
The comparison result of for the different values of α at $t = 1$.

Scheme	h	$\alpha = 1.3$		$\alpha = 1.6$	
		$E(h,\tau)$	Ord_1	$E(h,\tau)$	Ord_1
T-SAV	1/4	1.11e−3	–	1.80e−3	–
	1/8	1.13e−4	3.3024	1.82e−4	3.3454
	1/16	7.09e−6	3.9938	1.26e−5	3.8522
	1/32	4.50e−7	3.9745	8.14e−7	3.9498
	1/64	2.71e−8	4.0554	4.91e−8	4.0500
SAV	1/4	7.26e−2	–	8.85e−2	–
	1/8	1.66e−2	2.1250	2.01e−2	2.1388
	1/16	4.41e−3	1.9169	4.64e−3	2.1115
	1/32	1.49e−3	1.5594	1.08e−3	2.0996
	1/64	4.17e−4	1.8441	2.56e−4	2.0812
Linear-Implicit	1/4	3.51e−3	–	4.32e−3	–
	1/8	7.95e−4	2.1424	8.97e−4	2.2679
	1/16	1.98e−4	2.0055	2.24e−4	2.0016
	1/32	4.97e−5	1.9942	5.61e−5	1.9974
	1/64	1.23e−5	2.0146	1.42e−5	1.9821

Table 4
Discrete energy E^n for different values of α at different times t.

t	$\alpha = 1.2$	$\alpha = 1.5$	$\alpha = 1.8$	$\alpha = 2.0$
0	3.319216214209640	4.683481252241875	6.901017658923674	9.124413226435424
10	3.319216213476719	4.683481251241374	6.901017658950430	9.124413226532425
20	3.319216214426772	4.683481251054013	6.901017658429924	9.124413226771185
30	3.319216214485132	4.683481251783468	6.901017658114058	9.124413227012433
40	3.319216215210285	4.683481251096453	6.901017658280851	9.124413227123945
50	3.319216214460822	4.683481250647293	6.901017658768502	9.124413226755323

Example 5.2. Consider the problem with unknown exact solution as follows

$$u_{tt} + (-\Delta)^{\alpha/2}u + iu_t + |u|^2u = 0, \quad x \in [-5, 5], \ t \in [0, T]. \tag{5.2}$$

The initial conditions are selected as $u(x, 0) = (1 + i)x \exp(-10(1 - x)^2)$ and $u_t(x, 0) = 0$.

To verify the energy-conserving of the difference scheme (3.6)–(3.8), we calculate the values of the discrete energy E^n for different values of α at different times t, see Table 4. It is easy to see from that the T-SAV scheme (3.6)–(3.8) maintains the discrete energy well.

Also, the evolution of discrete energy E^n over a longer time interval ($T = 500$) for different values of α are depicted in Fig. 1 which indicates that the T-SAV scheme (3.6)–(3.8) captures the phenomenon of energy conservation, and it is suitable for long-term simulation.

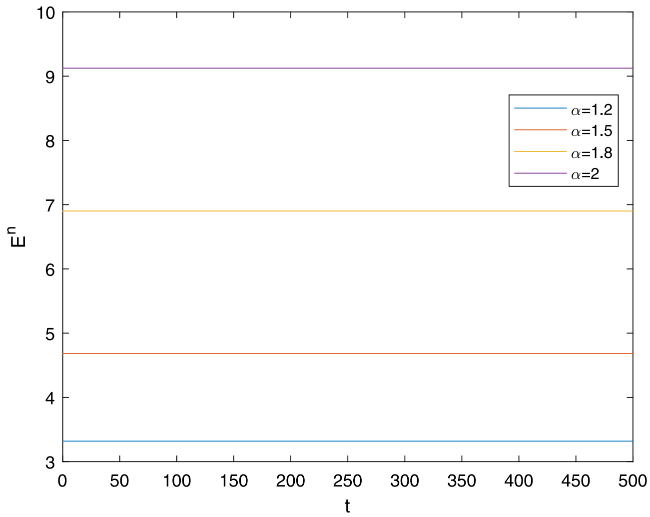

Fig. 1. The evolution of discrete energy E^n over time t for different values of α.

6. Conclusion

In this paper, based on T-SAV approach, we proposed and analyzed the higher order energy-preserving difference scheme for nonlinear space fractional Schrödinger equation with wave operator. It is proved that the solutions of the difference scheme are energy-preserving, bounded, and convergent in maximum norm. Finally, numerical examples for two fractional models illustrated that the proposed scheme can guarantee energy conservation of the system and has accuracy of 4 in space and 2 in time. It should be noted that, as far as we know, there is no theoretical support that the cosine value in Eq. (2.2) is always greater than 0, although the results based on various numerical examples so far show that this treatment is successful.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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